APPLICATION OF THE IMPROVED INTEGRAL METHOD OF LINES TO BOUNDARY-VALUE PROBLEMS IN HEAT CONDUCTION

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We discuss a generalization of the improved integral method of lines for the solution of the heat equation with boundary conditions of the second and third kinds.

An improved integral method of lines was given in [1] for the solution of the heat equation with boundary conditions of the first kind [2]. The improvement over the original method was obtained by integrating the heat equation along only part of partitioning interval with a certain weighting coefficient α, rather than along the entire interval. An additional algebraic equation was derived for this coefficient and the accuracy of the approximate solution was increased by this device by no less than two orders of magnitude; in certain cases the approximate solution coincides with the exact solution of the problem. When $\alpha = 0$ this method is completely equivalent to the spline method of [3], while for $\alpha^2 = 0.5$ it reduces to the improved method of lines [4], and for $\alpha = 1$ it reduces to the integral method of lines [5].

We extend the improved integral method of lines to the case of the heat equation with boundary conditions of the second and third kinds. The high intrinsic accuracy of this method allows one to reduce significantly the number of partitions. It is shown that for certain examples the approximate solution reduces to the exact one when the interval is partitioned into two arbitrary unequal parts.

Let it be required to find the solution of the heat equation subject to boundary conditions of the third kind:

$$U_t = a^2 U_{xx} + f(x,\ t) \quad (0 < x < 1,\ t > 0), \tag{1}$$

$$U(x,\ 0) = \varphi(x) \quad (0 \leqslant x \leqslant 1), \tag{2}$$

$$[\beta_1 U_x + \gamma_1 U]_{x=0} = \psi_1(t), \quad [\beta_2 U_x + \gamma_2 U]_{x=1} = \psi_2(t). \tag{3}$$

An approximate solution of the problem will be sought in the form of a polynomial

$$U(x,\ t) \approx \mathscr{P}(x,\ x_k,\ t) = \sum_{i=0}^{2} A_i^k (x - x_k)^i, \tag{4}$$

constructed on the uniform intervals

$$\Delta_x = \delta = 1/(N+1).$$

For simplicity we put $a = 1$. Following the improved integral method of lines [1] we integrate (1) on the interval $[x_k - \alpha_k\delta,\ x_k + \alpha_k\delta]$ using the approximate solution (4), where α_k is a weighting coefficient. We thereby obtain a system of N linear ordinary differential equations for A_0^k and A_2^k:

$$\dot{A}_0^k + \frac{1}{3}\alpha_k \delta_k^2 \dot{A}_2^k = 2A_2^k + \frac{1}{2\alpha_k\delta} \int\limits_{x_k - \alpha_k\delta}^{x_k + \alpha_k\delta} f(x,\ t)\, dx. \tag{5}$$

In order to solve this system of equations we rewrite the initial and boundary conditions in the form

$$\mathscr{P}(x_k,\ 0) = U(x_k,\ 0), \tag{6}$$

$$\beta_1 \mathscr{P}_x(0,\ x_1,\ t) + \gamma_1 \mathscr{P}(0,\ x_1,\ t) = \psi_1(t), \qquad \beta_2 \mathscr{P}_x(1,\ x_N,\ t) + \gamma_2 \mathscr{P}(1,\ x_N,\ t) = \psi_2(t). \tag{7}$$

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Equations (6) and (7) must be supplemented by the continuity condition on the temperature on the boundaries of the intervals

$$\mathcal{P}(x_k \pm \delta,\ x_k,\ t) = \mathcal{P}(x_{k\pm 1},\ x_{k\pm 1},\ t). \tag{8}$$

Substitution of the approximate solution (4) into (7) and (8) leads to the following matrix equation of order 2N for A_1^k and A_2^k:

$$By = Z, \tag{9}$$

where

$$y = [A_1^1,\ A_2^1,\ \ldots,\ A_1^N,\ A_2^N]^T,\quad Z = [\psi_1 - A_0^1,\ A_0^2 - A_0^1,\ \ldots,\ \psi_2 - A_0^N]^T. \tag{10}$$

Solving (9), we find expressions for A_1^k and A_2^k in terms of A_0^k, ψ_1 and ψ_2, We present the recursion relations for the A_2^k, since the A_1^k do not appear in (5):

$$\begin{gathered} A_2^1 = \frac{-\psi_1\delta - A_0^1(\beta_1 - 2\gamma_1\delta) + A_0^2(\beta_1 - \gamma_1\delta)}{\delta^2(3\beta_1 - 2\gamma_1\delta)}, \\ A_2^N = \frac{A_0^{N-1}(\beta_2 + \gamma_2\delta) - A_0^N(\beta_2 + 2\gamma_2\delta) + \psi_2\delta}{\delta^2(3\beta_2 + 2\gamma_2\delta)}, \\ A_2^k = (A_0^{k-1} - 2A_0^k + A_0^{k+1})/2\delta^2,\quad k = \overline{2,\ N-1}. \end{gathered} \tag{11}$$

With the help of (11), Eq. (5) can then be transformed to a matrix equation of order N:

$$\dot{A}_0 = CA_0 + D\psi + E\dot{\psi} + \Phi f, \tag{12}$$

where C, D, E, and Φ are variable matrices in t.

An analytical solution of (12) and (6) is written as

$$A_0 = \tilde{A}(t,\ 0)\varphi + \int\limits_0^t \tilde{A}(t,\ 0)\tilde{A}^{-1}(\tau,\ 0)(D\psi + E\dot{\psi} + \Phi f)\,d\tau. \tag{13}$$

It remains to find the weighting coefficient α_k. As in [1], we require that the approximate solution (4) satisfy Eq. (1) at t = 0 and $x = x_k$ (where the initial condition $U(x_k,\ 0) = \phi(x_k)$ is taken into account):

$$\dot{A}_0^k(0) = \varphi_{xx}(x_k) + f(x_k,\ 0),\quad k = \overline{1,\ N}. \tag{14}$$

Substituting $\dot{A}_0^k$ from (14) into (12) at t = 0, we obtain the following equation for the coefficient α_k:

$$C\varphi + D\psi + E\dot{\psi} + \Phi f = \varphi_{xx} + f. \tag{15}$$

We find α_k from (15) and then substitute into (13). Finally we thereby obtain the improved approximate solution at the points x_k.

When $\gamma_1 = \gamma_2 = 0$ the problem involves boundary conditions of the second kind. The reasoning in this case is similar to that given above.

The high intrinsic accuracy of the improved integral method of lines means that we can partition the entire interval up into a small number of parts. We show this for the case of boundary conditions of the second kind. A single partition point is used, which divides the whole interval into two equal or unequal parts.

Suppose it is required to find the time dependence of the temperature at a certain point $x_k = \Delta$. Using (4) we integrate (1) with respect to x on the interval $[\Delta - \alpha\Delta,\ \Delta + \alpha(1 - \Delta)]$. As a result we obtain a linear ordinary differential equation for the coefficients A_1, A_2, and A_0:

$$\dot{A}_0 + \frac{\alpha}{2}\dot{A}_1(1 - 2\Delta) + \frac{\alpha^2}{3}(1 - 3\Delta + 3\Delta^2)\dot{A}_2 = 2A_2 + \frac{1}{\alpha}\int\limits_{\Delta-\alpha\Delta}^{\Delta+\alpha(1-\Delta)} f(x,\ t)\,dx. \tag{16}$$

For simplicity we put f(x, t) = 0. Using (6) and (7) with $\gamma_1 = \gamma_2 = 0$ we find the coefficients A_1 and A_2 in terms of A_0 and ψ:

$$A_1 = (\psi_1\beta_2(1-\Delta) + \beta_1\psi_2\Delta)/\beta_1\beta_2, \quad A_2 = (\psi_2\beta_1 - \psi_1\beta_2)/2\beta_1\beta_2. \tag{17}$$

Substituting (17) into (16) we obtain a first-order differential equation for A_0:

$$\dot{A}_0 = L(\psi_1, \psi_2, \beta_1, \beta_2, \alpha, \Delta). \tag{18}$$

An analytical solution for (18) has the form

$$A_0 = A_0(0) + \int_0^t L d\tau. \tag{19}$$

In order to determine α we use (14) and (18) at t = 0. After substituting α into (19), we find an improved approximate solution at the point Δ.

The discussion is analogous for the determination of the temperature at the point Δ when boundary conditions of the third kind are specified.

In certain cases the method gives the exact solution of the problem. For example, this occurs in the solution of the following boundary-value problem for a single point Δ:

$$U_t = U_{xx}, \quad U(x, 0) = \sin\omega x,$$
$$U_x(0, t) = \omega\exp(-\omega^2 t), \quad U_x(1, t) = \omega\cos\omega\exp(-\omega^2 t).$$

Here the approximate solution reduces to the exact solution, which can be represented in the form:

$$U = \exp(-\omega^2 t)\sin\omega\Delta.$$

When boundary conditions of the third kind are specified, we have for the same point Δ:

$$U_t = U_{xx}, \quad U(x, 0) = \sin\omega x,$$
$$[U_x + U]_{x=0} = \omega\exp(-\omega^2 t), \quad [U_x + U]_{x=1} = (\omega\cos\omega + \sin\omega)\exp(-\omega^2 t),$$

and here the solution also coincides with the exact solution and has the form:

$$U = \exp(-\omega^2 t)\sin\omega\Delta.$$

NOTATION

x, linear coordinate; t, time; a, thermal diffusivity; f(x, t), heat source; U, temperature; Δ_x, position of the nodal point along the X axis; N, number of partitions along the X axis; β_1, γ_1, β_2, γ_2, given functions of time; $\tilde{A}(t, 0)$, fundamental solution matrix of the linear homogeneous system of differential equations.

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